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The Fourier Pseudospectral Method with a Restrain Operator for the RLW Equation

BEN-YU GUO AND WEI-MING CAO

Shanghai University of Science and Technology, Shanghai, China

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In this paper we develop a new Fourier pseudospectral method with a restrain operator which is applied to the RLW equation. The numerical results show the advantages of this method. We prove the generalized stability and the convergence of the scheme.

I. INTRODUCTION

In this paper we consider the periodical problem of the RLW equation

$$\begin{aligned} &\frac{\partial U}{\partial t}+\alpha\frac{\partial U}{\partial x}+U\frac{\partial U}{\partial x}-\delta\frac{\partial^3 U}{\partial t\,\partial x^2}=f, && x\in\vartheta;\ t\in(0,T],\\ &\frac{\partial^q U(0,t)}{\partial x^q}=\frac{\partial^q U(2,t)}{\partial x^q}, && q=0,1;\ t\in[0,T],\\ &U(x,0)=U_0(x), && x\in\vartheta, \end{aligned} \tag{1}$$

where $\vartheta=(0,2)$, $\delta>0$, $f(0,t)=f(2,t)$, and α is a constant.

In 1966, Peregrine [1] proposed the first numerical method for solving (1). In 1976, Abdulloev, Bogolubsky, Makhankov [2] showed numerically that the collision between two soliton-like waves is inelastic. Olver [3] proved that a RLW equation possesses only three conservations.

Other numerical methods can be found in Eilbeck and McGuire [4,5], Alexander and Morris [6], and Arnold *et al.*, [7]. Recently Wu and Guo [8], using a difference scheme with high accuracy, found that four waves were formed after the collision between two soliton-like waves. The accuracy of numerical solution of both finite element and difference methods is limited, even if the solution of (1) is very smooth.

The spectral method and pseudospectral method are two important numerical methods for P.D.E. (see Gottlieb and Orszag [9]). Pasciak [10] and Guo and Manoranjan [11] used these methods to solve the RLW equation. The pseudospectral method is preferable when dealing with nonlinear terms. However, it is less stable than the spectral method due to the aliasing interaction. For remedying this

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deficiency Kreiss and Oliger [12], Kuo [13], and Ma and Guo Ben-yu [14] proposed several filtering techniques. In this paper, we modify the filtering technique used in [14] and give a new pseudospectral method applied to (1). We test numerically the effect of the restrain operator and find that

(i) If the vibration of the solution of (1) is small and δ large, then the effect of the restrain operator is not clear.

(ii) If the vibration of the solution of (1) is big or δ very small, then the restrain operator increases the stability.

(iii) If we use the filtering technique in the same way as in [14] (i.e., scheme (8) in this paper), then the numerical result is poor, especially for very small δ (see Table VI, in Section III). But the new method proposed here gives a uniformly good approximation independent of δ (see Tables V, VI, in Section III).

We use the technique of generalized stability of Guo [15, 16] (i.e., g-stability; see Griffiths [17]), to strictly estimate the error. Four estimates are established. Only two of them depend on δ. The convergence is given also.

II. The Scheme

Let

$$(v, w) = \int_0^2 v(x)\,\overline{w(x)}\,dx,$$

$$\|v\|^2 = (v, v), \qquad |v|_1 = \left\|\frac{\partial v}{\partial x}\right\|.$$

Now we derive the new pseudospectral method. Let N be a positive integer and n any integer:

$$V_n = \mathrm{Span}\{e^{n\pi i x}/|n| \leqslant N\}.$$

$\dot{V}_N$ is a real value function subset of V_N. Let P_N be the orthogonal projection operator from $L^2(\vartheta)$ to V_N, i.e.,

$$(p_N v, w) = (v, w), \qquad \forall w \in V_N.$$

Let P_c be the interpolation operator from $C(\mathcal{9})$ to V_N such that

$$p_c v(x_j) = v(x_j), \qquad x_j = \frac{2j}{2N+1}, \qquad j = 0, 1, \ldots, 2N.$$

From [12],

$$(v, w)_N = (p_c v, p_c w)_N = (p_c v, p_c w), \qquad \forall v, w \in c(\mathcal{9}), \tag{2}$$

where

$$(v, w)_N = \frac{2}{2N+1} \sum_{j=0}^{2N} v(x_j)\, \overline{w(x_j)}.$$

Let τ be the mesh spacing of the variable t, $S_\tau = \{k \mid 0 \leqslant k \leqslant [T/\tau] - 1\}$ and $v^k(x) = v(x, k\tau)$. Define

$$v_t^k(x) = \frac{1}{\tau}\,(v^{k+1}(x) - v^k(x)).$$

As is well known, a reasonable scheme must keep the properties similar to those of (1). Indeed the solution of (1) satisfies the following conservations

$$\int_0^2 U(x, t)\, dx = \int_0^2 U_0(x)\, dx + \int_0^t \int_0^2 f(x, \xi)\, dx\, d\xi. \tag{3}$$

$$\|U(t)\|^2 + \delta\, |U(t)|_1^2 = \|U_0\|^2 + \delta\, |U_0|_1^2 + 2 \int_0^t (f(\xi), U(\xi))\, d\xi. \tag{4}$$

In order to simulate (3) and (4), we define the following operator

$$J(v(x), \varphi(x)) = \frac{1}{3} p_c\left(\varphi(x) \frac{\partial}{\partial x} v(x)\right) + \frac{1}{3} \frac{\partial}{\partial x} p_c(\varphi(x), v(x)).$$

It can be shown that for all v, w, $\varphi \in \dot{V}_N$,

$$(J(v, \varphi), \omega) + (J(w, \varphi), v) = 0. \tag{5}$$

The simplest pseudospectral method for (1) is to find $u^k \in \dot{V}_N$ such that

$$\begin{aligned} &u_t^k(x) + \alpha \frac{\partial}{\partial x} u^k(x) + J(u^k(x) + \sigma\tau u_t^k(x), u^k(x)) - \delta \frac{\partial^2}{\partial x^2} u_t^k(x) \\ &\quad = p_c f^k(x), \qquad x \in \vartheta,\ k \in s_\tau, \\ &u^0(x) = P_c U_0(x), \qquad x \in \vartheta, \end{aligned} \tag{6}$$

where $0 \leqslant \delta \leqslant 1$. But the numerical results showed that scheme (6) is not stable for very small δ. On the other hand, for the error estimation uniformly for δ, we have to establish the following inequality

$$\left|\left(F\left(v \frac{\partial v}{\partial x}\right), w\right)_N\right| \leqslant b(w)\, \|v\|^2, \qquad \forall v, w \in V_N, \tag{7}$$

where $b(w)$ is independent of N, and

$$F = \left(I - \delta \frac{\partial^2}{\partial x^2}\right)^{-1}.$$

But the inequality (7) is not true generally. For instance, we consider the following functions

$$v(x) = a_N\, e^{N\pi ix}, \qquad w(x) = 1 + 2i \sin \pi x,$$

and so from (2)

$$\left(F\left(v\frac{\partial v}{\partial x}\right),\ w\right)_N = \left(p_c F\left(v\frac{\partial v}{\partial x}\right), w\right).$$

Since

$$\begin{aligned}\left|\left(p_c F\left(v\frac{\partial v}{\partial x}\right), w\right)\right| &= \frac{\pi N a_N^2}{1+\delta\pi^2 N^2}\ |(P_c\, e^{2N\pi ix}, w)| \\ &= \frac{\pi N a_N^2}{1+\delta\pi^2 N^2}\ |(P_c\, e^{-\pi ix}, w)| \\ &= \frac{\pi N a_N^2}{1+\delta\pi^2 N^2}\ |(e^{-\pi ix}, w)| \\ &= \frac{2\pi N a_N^2}{1+\delta\pi^2 N^2},\end{aligned}$$

the constant $b(w)$ is much larger for very small δ and large N.

Obviously the main difficulties in the previous paragraph come from the high frequency terms. In order to remedy it, the restrain operator R_ν is proposed in [13, 14] such that for

$$v(x) = \sum_{|n| \leqslant N} a_n\, e^{n\pi ix},$$

we have

$$R_\nu v(x) = \sum_{|n| \leqslant N} \left(1 - \left|\frac{n}{N}\right|^\nu\right) a_n\, e^{n\pi ix}.$$

In [14], the nonlinear term $U(x, t)(\partial/\partial x)\ U(x, t)$ is approximated by $J(u^k(x) + \sigma\tau u_t^k(x),\ R_\nu u^k(x))$, which is successfully applied to the KDV equation. If we generalize this method to (1) directly, then we have

$$\begin{aligned}&u_t^k(x) + \alpha\frac{\partial}{\partial x} u^k(x) + J(u^k(x) + \sigma\tau u_t^k(x),\ R_\nu u^k(x)) - \delta\frac{\partial^2}{\partial x^2} u_t^k(x) \\ &\qquad = p_c f^k(x), \qquad x \in \vartheta,\ k \in S_\tau, \\ &u^0(x) = p_c U_0(x), \qquad x \in \vartheta. \qquad (8)\end{aligned}$$

When $\sigma=0$, we can solve (10) explicitly. This is one of the advantages of the spectral method and pseudospectral methods. But the numerical result is unsatisfactory. In particular, the error is very big for $\alpha\neq 0$ and small δ (see Table VI, in Section III).

In this paper, we approximate the nonlinear term by $R_v J(R_v u^k(x)+\sigma\tau R_v u_t^k(x), u^k(x))$ and use the filtering technique for the linear term $\alpha(\partial/\partial x)\, u^k(x)$ and the right term $p_c f^k(x)$. Now we have the scheme

$$
\begin{aligned}
&u_t^k(x)+\alpha R_v \frac{\partial}{\partial x} u^k(x)+R_v J(R_v u^k(x)+\sigma\tau R_v u_t^k(x), u^k(x)) \\
&\qquad -\delta \frac{\partial^2}{\partial x^2} u_t^k(x)=R_v p_c f^k(x), \qquad x\in \vartheta. \qquad k\in s_\tau, \qquad (9)\\
&u^0(x)=p_c U_0(x), \qquad x\in\vartheta.
\end{aligned}
$$

Clearly,

$$\left(R_v \frac{\partial}{\partial x} v, v\right)=\left(\frac{\partial}{\partial x} R_v^{1/2} v, R_v^{1/2} v\right)=0, \qquad \forall v\in V_N. \tag{10}$$

From (5), for all v, w, $\varphi\in \dot{V}_N$,

$$(R_v J(R_v v, \varphi), w)+(R_v J(R_v w, \varphi), v)=0. \tag{11}$$

If $\sigma=\frac{1}{2}$, then it follows from (9)–(11) that

$$
\begin{aligned}
&\int_0^2 u^k(x)\,dx=\int_0^2 u^0(x)\,dx+\tau\sum_{\xi=0}^{k-1}\int_0^2 R_v p_c f^\xi(x)\,dx,\\
&\|u^k\|^2+\delta\,|u^k|_1^2=\|u^0\|^2+\delta\,|u^0|_1^2+\tau\sum_{\xi=0}^{k-1}(R_v p_c f^\xi, u^\xi+u^{\xi+1}),
\end{aligned}
$$

which are the discrete analogy of (3) and (4).

III. Numerical Results

We test the effect of the restrain operator with the function

$$U(x,t)=A\exp(A\sin\pi x+0.5t).$$

Put $N=8$, $\tau=0.01$, and let

$$E(t)=\max_{0\leqslant j\leqslant 2N}\frac{|u(x_j,t)-U(x_j,t)|}{|U(x_j,t)|}.$$

All computations are carried out with the explicit scheme, i.e., $\sigma = 0$. The numerical results show that

(i) If the vibration of the genuine solution of (1) is small (e.g., $A = 0.1$) and $\alpha = 0$, then the effect of R_v is not clear (see Table I). The accuracy of (9) is nearly the same as the spectral method of [11].

(ii) If δ is large (e.g., $\delta = 1$), then the operator R_v is not important even though the vibration of a genuine solution is big (see Table II).

(iii) If the vibration of a genuine solution is big and δ small (e.g., $A = 1$, $\delta = 10^{-6}$, 10^{-4}), then the effect of R_v is very clear, see Tables III and IV.

(iv) The smaller the parameter δ, the more important the linear term $\alpha(\partial U/\partial x)$. Thus the operator R_v is very important even if the vibration of the genuine solution is not big. Tables I and IV show that the effect of R_v is not clear for $\alpha = 0$, $A = 0.1$, but very clear for $\alpha = 2$, $A = 0.1$.

(v) The value of v in the restrain operator must be suitably chosen. If v is too large, the filtering technique is weakened. If v is too small, the approximation accuracy is lowered. The suitable value of v is between 5 and 10. But the best value of v is different in different cases. For instance, the best choice in Table IV is $v = 3$.

(vi) For the proof of convergence, the filtering technique for the linear term and right term is not important. But numerical experiment shows the importance, see Table V. In Table V, schemes $(9)_{a\text{-}c}$ are similar to (9), but the linear term and the right term are respectively approximated in the following

$$\text{Scheme (9)a:} \qquad \alpha \frac{\partial}{\partial x} u^k(x), \qquad R_v p_c f^k(x);$$

$$\text{Scheme (9)b:} \qquad \alpha R_v \frac{\partial}{\partial x} u^k(x), \qquad p_c f^k(x);$$

$$\text{Scheme (9)c:} \qquad \alpha \frac{\partial}{\partial x} u^k(x), \qquad p_c f^k(x).$$

(vii) The numerical results of scheme (8) are much worse than that of scheme (9) and so the new filtering technique in this paper is better for the explicit scheme of a nonlinear problem. In Table VI, schemes (8) a–c are similar to scheme (8) with the linear term and the right term approximated by

$$\text{Scheme (8)a:} \qquad \alpha R_v \frac{\partial}{\partial x} u^k(x), \qquad p_c f^k(x);$$

$$\text{Scheme (8)b:} \qquad \alpha \frac{\partial}{\partial x} u^k(x), \qquad R_v p_c f^k(x);$$

$$\text{Scheme (8)c:} \qquad \alpha \frac{\partial}{\partial x} u^k(x), \qquad p_c f^k(x).$$

TABLE I

Scheme (9), $\sigma = 0$, $\alpha = 0$, $A = 0.1$.

$E(5.0)$	$v = 5$	$v = 10$	$v = 50$
$\delta = 10^{-6}$	0.00228242	0.00228203	0.00228193
$\delta = 10^{-4}$	0.00228399	0.00228281	0.00228183
$\delta = 10^{-2}$	0.00229817	0.00229622	0.00229651
$\delta = 1$	0.00236857	0.00236493	0.00236557

TABLE II

Scheme (9), $\sigma = 0$, $\delta = 1$, $A = 1$.

$E(5.0)$	$v = 5$	$v = 10$	$v = 50$
$\alpha = 2$	0.00144118	0.00118726	0.00115333
$\alpha = 0$	0.00231011	0.00215183	0.00216537

TABLE III

Scheme (9), $\sigma = 0$, $\alpha = 0$, $A = 1$.

$E(1.0)$	$v = 5$	$v = 10$	$v = 50$
$\delta = 10^{-6}$	0.00329499	0.0355039	0.235286
$\delta = 10^{-4}$	0.0283234	0.0124055	0.170894

TABLE IV

Scheme (9), $\alpha = 2$, $A = 0.1$

$E(2.0)$	$v = 3$	$v = 5$	$v = 10$	$v = 50$
$\delta = 10^{-6}$	0.00156715	0.0186465	0.0243057	> 10
$\delta = 10^{-4}$	0.00155333	0.00163698	0.0111043	0.702116

TABLE V

$\sigma = 0$, $\alpha = 2$, $v = 10$, $\delta = 10^{-6}$

	$E(0.5)$, $A = 1$.	$E(2.0)$, $A = 0.1$.
Scheme (9)	0.0454317	0.0243057
Scheme (9)a	0.0745903	7.42054
Scheme (9)b	0.564465	0.0275196
Scheme (9)c	2.08395	9.66944

TABLE VI

$\sigma = 0, \alpha = 2, A = 1, \nu = 10, \delta = 10^{-6}$

	Scheme (8)	Scheme (8)a	Scheme (8)b	Scheme (8)c
$E(0.5)$	0.136626	0.271632	7.86826	2.89614

(viii) Another pseudospectral scheme is

$$u_t^k(x) + \alpha R_\nu \frac{\partial}{\partial x} u^k(x) + R_\nu J(R_\nu u^k(x) + \sigma\tau R_\nu u_t^k(x), R_\nu u^k(x)) - \delta \frac{\partial^2}{\partial x^2} u_t^k(x) = R_\nu p_c f^k(x). \quad (12)$$

This scheme seems more reasonable, but the numerical results show that the accuracy of (12) is nearly the same as scheme (9).

TABLE VII

$\sigma = 0, A = 1, \nu = 10$

	$E(0.5), \alpha = 2$		$E(1.0), \alpha = 0$	
	$\delta = 10^{-6}$	$\delta = 10^{-4}$	$\delta = 10^{-6}$	$\delta = 10^{-4}$
Scheme (9)	0.0454317	0.0311898	0.0355039	0.0124055
Scheme (12)	0.0586913	0.0368031	0.0129624	0.0181770

TABLE VIII

$\sigma = 0, A = 0.1, \nu = 10$

	$E(2.0), \alpha = 2$		$E(5.0), \alpha = 0$	
	$\delta = 10^{-6}$	$\delta = 10^{-4}$	$\delta = 10^{-6}$	$\delta = 10^{-4}$
Scheme (9)	0.0243057	0.0111043	0.00228203	0.00228281
Scheme (12)	0.0244925	0.0124284	0.00228242	0.00228232

IV. Error Estimations

Assume that $u^0(x)$ and $f^k(x)$ have respectively the error $\tilde{u}^0(x)$ and $\tilde{f}^k(x)$. Then the error of $u^k(x)$, denoted by $\tilde{u}^k(x)$, satisfies

$$\tilde{u}_t^k(x)+\alpha R_v \frac{\partial}{\partial x}\tilde{u}^k(x)+R_v J(R_v\tilde{u}^k(x)+\sigma\tau R_v\tilde{u}_t^k(x), \tilde{u}^k(x)+u^k(x))$$
$$+R_v J(R_v u^k(x)+\sigma\tau R_v u_t^k(x), \tilde{u}^k(x))-\delta\frac{\partial^2}{\partial x^2}\tilde{u}_t^k(x)=R_v p_c \tilde{f}^k(x). \qquad (13)$$

Let

$$L^p(\vartheta)=\left\{v \mid \|v\|_{L^p}=\left(\int_0^2 |v(x)|^p\,dx\right)^{1/p}<\infty\right\}.$$

In particular, the inner product and the norm are $(\cdot,\cdot)$ and $\|\cdot\|$, respectively, for $p=2$. For any positive integer, let $|v|_\beta=\|\partial^\beta u/\partial x^\beta\|$,

$$H^\beta(\vartheta)=\left\{v \,\middle|\, \|v\|_\beta=\sum_{q=1}^{\beta}|v|_q<\infty\right\}.$$

For any positive constant β, $H^\beta(\vartheta)$ is the complex interpolation space between $H^{[\beta]}(\vartheta)$ and $H^{[\beta]+1}(\vartheta)$. Define

$$H_p^\beta(\vartheta)=\{v \mid v\in H^\beta(\vartheta),\ v(x)=v(x+2)\}.$$

We denote by $C^q(0,T;H_p^\beta(\vartheta))$ the space of abstract functions with the norm

$$\|v\|_{C^q(0,T;H_p^\beta(\vartheta))}=\max_{0\leqslant s\leqslant q}\max_{0\leqslant t\leqslant T}\left\|\frac{\partial^s v(t)}{\partial t^s}\right\|_\beta.$$

Similarly we define the space $H^q(0,T;H_p^\beta(\vartheta))$. Let $|||v|||_\beta=\max_{0\leqslant k\tau\leqslant T}\|v^k\|_\beta$. In particular, $|||v|||=\max_{0\leqslant k\tau\leqslant T}\|v^k\|$. Let p be a suitably small positive constant and

$$Q^k(v)=\|v^k\|^2+\delta\,|v^k|_1^2+p\tau^2\sum_{\xi=0}^{k-1}(\|v_t^\xi\|^2+\delta\,|v_t^\xi|_1^2),$$
$$\rho^k(v,w)=\|v\|^2+\delta\,|v|_1^2+\tau\sum_{\xi=0}^{k-1}\|w^k\|^2.$$

Theorem 1. (i) *If $\sigma\leqslant\frac{1}{2}$, then there exist positive constants b_1, b_2, b_3 depending only on $|||u|||_{3/2+\gamma}$ $(\gamma>0)$ and δ such that for all $k\tau\leqslant T$, $\rho^k(\tilde{u}^0,\tilde{f})\leqslant b_1/\tau N$,*

$$Q^k(\tilde{u})\leqslant b_2\rho^k(\tilde{u}^0,\tilde{f})\,e^{b_3 k\tau}.$$

(ii) *If* $\sigma > \frac{1}{2}$ *then there exist positive constants* b_4, b_5 *depending on* $|||u|||_{3/2+\gamma}$ *and* δ *such that for all* k *and* $\rho^k(\tilde{u}^0, \tilde{f})$,

$$Q^k(\tilde{u}) \leqslant b_4 \rho^k(\tilde{u}^0, \tilde{f})\, e^{b_5 k\tau}.$$

(iii) If $\sigma \leqslant \frac{1}{2}$, $\tau = O(N^{-2})$, *then there exist positive constants* b_6, b_7, b_8 *depending only on* $|||u|||_{3/2+\gamma}$ *such that for all* $k\tau \leqslant T$, $\rho^k(\tilde{u}^0, \tilde{f}) \leqslant b_6 N^{-1}$,

$$Q^k(\tilde{u}) \leqslant b_7 \rho^k(\tilde{u}^0, \tilde{f})\, e^{b_8 k\tau}.$$

(iv) *If* $\sigma > \frac{1}{2}$, $\tau = O(N^{-2})$, *then there exist positive constants* b_9, b_{10} *depending only on* $|||u|||_{3/2+\gamma}$ *such that for all* k *and* $\rho^k(\tilde{u}^0, \tilde{f})$,

$$Q^k(\tilde{u}) \leqslant b_9 \rho^k(\tilde{u}^0, \tilde{f})\, e^{b_{10} k\tau}.$$

Next consider the convergence. Let $U(x, t)$ and $u^k(x)$ be the solutions of (1) and (9), respectively.

THEOREM 2. *Assume that*

(i) $U \in H^2(0, T; H_p^1(\vartheta)) \cap C(0, T; H_p^\beta(\vartheta))$, $U_0 \in H_p^\beta(\vartheta)$, $f \in C(0, T; H_p^{\beta-1}(\vartheta))$;

(ii) $v \geqslant \beta \geqslant 2$,

then for all $k\tau \leqslant T$,

$$Q^k(u - U) \leqslant b_{11} e^{b_{12} k\tau} (\tau^2 + N^{2-2\beta}),$$

where b_{11} *and* b_{12} *are positive constants depending on* $\|U_0\|_\beta$, $\|f\|_{C(0, T; H^{\beta-1}(\vartheta))}$, $\|U\|_{H^2(0, T; H^1(\vartheta))}$, $\|U\|_{C(0, T; H^\beta(\vartheta))}$, *and* δ. *If* $\tau = O(N^{-2})$, *then* b_{11} *and* b_{12} *are independent of* δ.

V. SOME LEMMAS

For the proof of the theorems, we need the following lemmas.

LEMMA 1 [12]. *If* $0 \leqslant \mu \leqslant \beta$, $v \in H_p^\beta(\vartheta)$, then

$$\|P_N v - v\|_\mu \leqslant c N^{\mu-\beta} |v|_\beta.$$

If $\beta > \frac{1}{2}$, in addition, then

$$\|P_c v - v\|_\mu \leqslant c N^{\mu-\beta} |v|_\beta.$$

LEMMA 2 [12]. *If* $0 \leqslant \mu \leqslant \beta$, $v \in V_N$, *then*

$$\|v\|_\beta \leqslant c N^{\beta-\mu} \|v\|_\mu.$$

LEMMA 3 [12]. *If* $v, w \in V_N$, *then*

$$\|vw\|^2 \leqslant (2N+1)\,\|v\|^2\,\|w\|^2.$$

LEMMA 4 [14]. *If* $0 \leqslant \mu \leqslant \beta \leqslant \nu$, $v \in V_N$, *then*

$$\|R_\nu v - v\|_\mu \leqslant cN^{\mu-\beta}\,|v|_\beta.$$

For simplicity, we introduce the circle convolution operator "$*$" such that if

$$v(x) = \sum_{|n| \leqslant N} a_n\, e^{n\pi i x}, \qquad w(x) = \sum_{|n| \leqslant N} b_n\, e^{n\pi i x},$$

then

$$v * w(x) = \sum_{|n| \leqslant N} \sum_{|l| \leqslant N} a_l b_{n-l}\, e^{n\pi i x},$$

where $a_{n+2N+1} = a_n$, $b_{n+2N+1} = b_n$. From [14], for $v, w \in V_N$, $\varphi \in \dot{V}_N$, we have

$$P_c(v(x)\,w(x)) = v * w(x), \tag{14}$$

$$(v * \varphi, w) = (v, \varphi * \omega). \tag{15}$$

LEMMA 5 [14]. *If* $v \in V_N$, $w \in H_p^{3/2+\gamma}(\vartheta)$, $\gamma > 0$, *then*

$$\left|\left(R_\nu v * \frac{\partial v}{\partial x}, w\right)\right| \leqslant c_\gamma \nu\, \|w\|_{3/2+\gamma}\, \|v\|^2,$$

$$\left|\left(v * R_\nu \frac{\partial v}{\partial x}, w\right)\right| \leqslant c_\gamma \nu\, \|w\|_{3/2+\gamma}\, \|v\|^2,$$

where c_γ is a positive constant depending only on γ.

LEMMA 6. *If* $v, w \in V_N$, $\gamma > 0$, *then*

$$|(R_\nu J(R_\nu w, v), v)| \leqslant c_\gamma\, \|w\|_{3/2+\gamma}\, \|v\|^2.$$

Proof. We have

$$J(R_\nu w, v) = \frac{1}{3} v * R_\nu \frac{\partial w}{\partial x} + \frac{1}{3} \frac{\partial}{\partial x}(v * R_\nu w),$$

$$\begin{aligned} |(R_\nu J(R_\nu w, v), v)| &= |(J(R_\nu \omega, v), R_\nu v)| \\ &\leqslant \frac{1}{3}\left|\left(v * R_\nu \frac{\partial w}{\partial x},\ R_\nu v\right)\right| + \frac{1}{3}\left|\left(\frac{\partial}{\partial x}(v * R_\nu w), R_\nu v\right)\right|. \end{aligned}$$

From (15) and Lemma 5

$$\left|\left(v * R_\nu \frac{\partial w}{\partial x}, R_\nu v\right)\right| \leqslant c \left\|\frac{\partial w}{\partial x}\right\|_{L^\infty(\mathcal{I})} \|v\|^2 \leqslant c_\gamma \|w\|_{3/2+\gamma} \|v\|^2,$$

$$\left|\left(\frac{\partial}{\partial x}(v * R_\nu w), R_\nu v\right)\right| = \left|\left(v * R_\nu \frac{\partial v}{\partial x}, R_\nu w\right)\right| \leqslant c_\gamma \|w\|_{3/2+\gamma} \|v\|^2.$$

This completes the proof.

LEMMA 7 [15]. *If the following conditions are fulfilled*

(i) Z^k *is nonnegative function of* k, M_1, M_2, M_3, *and* ρ *are nonnegative constants,*

(ii) α_1, α_2, *and* α_3 *are constants,*

(iii) *if* $Y \leqslant M_1 N^{\alpha_1}$, *then* $\eta(Y) \leqslant 0$,

(iv) *for all* $k \geqslant 1$,

$$Z^k \leqslant \rho + \tau \sum_{\xi=0}^{k-1} [M_2 Z^\xi + M_3 (Z^\xi)^{\alpha_2} N^{\alpha_3} + \eta(Z^\xi)],$$

(v) $Z^0 \leqslant \rho \leqslant e^{-(M_2+M_3)T} \min(N^{-\alpha_3/\alpha_2}, M_1 N^{\alpha_1})$,

then for all $k\tau \leqslant T$,

$$Z^k \leqslant \rho\, e^{(M_2+M_3)k\tau}. \tag{16}$$

In particular, (16) *holds for all* k *and* ρ *provided* $M_3 = 0$ *and* $\eta(Y) \leqslant 0$.

VI. The Proof of Theorem 1

By taking the inner product of (13) with $2\tilde{u}^k(x)$, we have from (10) and (11) that

$$\begin{aligned}&(\|\tilde{u}^k\|^2 + \delta\, |\tilde{u}^k|_1^2)_t - \tau(\|\tilde{u}^k_t\|^2 + \delta\, |\tilde{u}^k_t|_1^2) - 2\sigma\tau(R_\nu J(R_\nu \tilde{u}^k,\\ &u^k + \tilde{u}^k),\, \tilde{u}^k_t) + 2(R_\nu J(R_\nu u^k + \sigma\tau R_\nu u^k_t, \tilde{u}^k), \tilde{u}^k)\\ &\quad = 2(R_\nu p_c \tilde{f}^k, \tilde{u}^k).\end{aligned} \tag{17}$$

Let m and ε be positive constants. Taking the inner product of (13) with $m\tau\tilde{u}^k_t$, it follows from (11) that

$$\begin{aligned}&m\tau(\|\tilde{u}^k_t\|^2 + \delta\,|\tilde{u}^k_t|_1^2) + \alpha m\tau\left(R_\nu \frac{\partial}{\partial x}\tilde{u}^k, \tilde{u}^k_t\right) + m\tau(R_\nu J(R_\nu \tilde{u}^k, u^k + \tilde{u}^k),\\ &\tilde{u}^k_t) + m\tau(R_\nu J(R_\nu u^k + \sigma\tau R_\nu u^k_t, \tilde{u}^k), \tilde{u}^k_t) = m\tau(R_\nu p_c \tilde{f}^k, \tilde{u}^k_t).\end{aligned} \tag{18}$$

The combination of (17) with (18) leads to

$$(\|\tilde{u}^k\|^2+\delta\,|\tilde{u}^k|_1^2)_t+\tau(m-1-\varepsilon)(\|\tilde{u}_t^k\|^2+\delta\,|\tilde{u}_t^k|_1^2)+\sum_{l=1}^{4}F_l^k$$
$$\leqslant\|\tilde{u}^k\|^2+\left(1+\frac{\tau m^2}{4\varepsilon}\right)\|p_c\tilde{f}^k\|^2, \tag{19}$$

where

$$F_1^k=\tau(m-2\sigma)(R_vJ(R_v\tilde{u}^k,u^k+\tilde{u}^k),\tilde{u}_t^k),$$
$$F_2^k=(R_v\,J(R_vu^k+\sigma\tau R_vu_t^k,\;\tilde{u}^k),\tilde{u}^k),$$
$$F_3^k=m\tau(R_v\,J(R_vu^k+\sigma\tau R_vu_t^k,\tilde{u}^k),\tilde{u}_t^k),$$
$$F_4^k=\alpha m\tau\left(R_v\frac{\partial}{\partial x}\tilde{u}^k,\tilde{u}_t^k\right).$$

Now we estimate $|F_l^k|$. From Lemmas 1–4, we have

$$|F_1^k|\leqslant\varepsilon\tau\,\|\tilde{u}_t^k\|^2+\frac{c\tau(m-2\sigma)^2}{\varepsilon}\left(\left\|\frac{\partial u^k}{\partial x}\right\|_{L^\infty(\vartheta)}^2\|\tilde{u}^k\|^2+\|u^k\|_{L^\infty(\vartheta)}|\tilde{u}^k|_1^2\right.$$
$$\left.+N\,\|\tilde{u}^k\|^2\,|\tilde{u}^k|_1^2\right)$$
$$\leqslant\varepsilon\tau\,\|\tilde{u}_t^k\|^2+\frac{c\tau(m-2\sigma)^2}{\varepsilon}(\|u^k\|_{3/2+\gamma}(\|\tilde{u}^k\|^2+|\tilde{u}^k|_1^2)+N\,\|\tilde{u}^k\|^2\,|\tilde{u}^k|_1^2).$$

Lemma 6 leads to

$$|F_2^k|\leqslant c\,|||u|||_{3/2+\gamma}\,\|\tilde{u}^k\|^2.$$

We have also

$$|F_3^k|\leqslant\varepsilon\tau\,\|\tilde{u}_t^k\|^2+\frac{c\tau m^2}{\varepsilon}\,|||u|||_{3/2+\gamma}^2(\|\tilde{u}^k\|^2+|\tilde{u}^k|_1^2),$$
$$|F_4^k|\leqslant\varepsilon\tau\,\|\tilde{u}_t^k\|^2+\frac{\tau\alpha^2m^2}{\varepsilon}\,|\tilde{u}^k|_1^2.$$

Substituting the above estimations into (19), we get

$$(\|\tilde{u}^k\|^2+\delta\,|\tilde{u}^k|_1^2)_t+\tau(m-1-4\varepsilon)(\|\tilde{u}_t^k\|^2+\delta\,|\tilde{u}_t^k|_1^2)\leqslant R^k, \tag{20}$$

where

$$R^k = c\left\{\left[1+\frac{\tau m^2}{\varepsilon}+\left(1+\frac{\tau(m-2\sigma)^2}{\varepsilon}\right)|||u|||^2_{3/2+\gamma}\right]\|\tilde{u}^k\|^2+\frac{\tau(m-2\sigma)^2+\tau m^2}{\varepsilon}\right.$$

$$\left.\times(|||u|||^2_{3/2+\gamma}+1)|\tilde{u}^k|_1^2+\frac{\tau N(m-2\sigma)^2}{\varepsilon}\|\tilde{u}^k\|^2\,|\tilde{u}^k|_1^2+\left(1+\frac{\tau m^2}{4\varepsilon}\right)\|\tilde{f}^k\|^2\right\}.$$

Let p be suitably small. Take $\varepsilon=\frac{1}{4}(m-1-p)>0$ and

$$m=2\sigma,\qquad H_\sigma=0,\qquad \text{for}\quad \sigma>\tfrac{1}{2},$$
$$m=2,\qquad H_\sigma=1,\qquad \text{for}\quad \sigma\leqslant\tfrac{1}{2}.$$

Then

$$(\|\tilde{u}^k\|^2+\delta\,|\tilde{u}^k|_1^2)_t+p\tau(\|\tilde{u}_t^k\|^2+\delta\,|\tilde{u}_t^k|_1^2)\leqslant R^k, \tag{21}$$

from which

$$Q^k(\tilde{u})\leqslant c\rho^k(\tilde{u}^0,\tilde{f})+c\tau\sum_{\xi=0}^{k-1}\left[(1+|||u|||^2_{3/2+\gamma})(\|\tilde{u}^\xi\|^2+\delta|\tilde{u}^\xi|_1^2)+\tau NH_\sigma\,\|\tilde{u}^\xi\|^2\,|\tilde{u}^\xi|_1^2\right].$$

Since Lemma 2 and

$$N\|\tilde{u}^k\|^2\,|\tilde{u}^k|_1^2\leqslant\delta N\,|\tilde{u}^k|_1^4+\frac{N}{4\delta}\,\|\tilde{u}^k\|^4$$

or

$$N\,\|\tilde{u}^k\|^2\,|\tilde{u}^k|_1^2\leqslant cN^3\,\|\tilde{u}^k\|^4.$$

Lemma 7 completes the proof.

VII. The Proof of Theorem 2

Let $W(x,t)=p_N U(x,t)$. From (1), we have

$$\frac{\partial W(x,t)}{\partial t}+\alpha\frac{\partial W(x,t)}{\partial x}+p_N\left(U(x,t)\frac{\partial U(x,t)}{\partial x}\right)-\delta\frac{\partial^3 W(x,t)}{\partial t\,\partial x^2}=p_N f(x,t)$$

and so

$$W_t^k(x)+\alpha R_\nu\frac{\partial W^k(x)}{\partial x}+R_\nu\,J(R_\nu W^k(x)+\sigma\tau R_\nu W_t^k(x),\,W^k(x))$$
$$-\delta\frac{\partial^2 W_t^k(x)}{\partial x^2}=R_\nu p_c f^k(x)+\sum_{l=1}^{5}g_l^k(x), \tag{22}$$

where

$$g_1^k(x) = W_t^k(x) - \frac{\partial W(x, k\tau)}{\partial t},$$

$$g_2^k(x) = \alpha R_v \frac{\partial W^k(x)}{\partial x} - \alpha \frac{\partial W^k(x)}{\partial x},$$

$$g_3^k(x) = R_v J(R_v W^k(x) + \sigma\tau R_v W_t^k(x), W^k(x)) - p_N \left(U(x, k\tau) \frac{\partial U(x, k\tau)}{\partial x} \right),$$

$$g_4^k(x) = \delta \frac{\partial^3 W^k(x)}{\partial t\, \partial x^2} - \delta \frac{\partial^2 W_t^k(x)}{\partial x^2},$$

$$g_5^k(x) = p_N f^k(x) - R_v p_c f^k(x).$$

Let $e^k(x) = u^k(x) - W^k(x)$. From (9) and (22), it follows that

$$e_t^k(x) + \alpha R_v \frac{\partial e^k(x)}{\partial x} + R_v J(R_v e^k(x) + \sigma\tau R_v e_t^k(x), W^k(x) + e^k(x)) + R_v J(R_v W^k(x)$$
$$+ \sigma\tau R_v W_t^k(x), e^k(x)) - \delta \frac{\partial^2 e_t^k(x)}{\partial x^2} = -\sum_{l=1}^{5} g_l^k(x).$$

Integration by parts leads to

$$\frac{\partial W(x, k\tau)}{\partial t} - W_t^k(x) = -\frac{1}{\tau} \int_{k\tau}^{(k+1)\tau} (k\tau + \tau - \eta) \frac{\partial^2 W(x, \eta)}{\partial \eta^2}\, d\eta$$

and thus

$$\tau \sum_{\xi=0}^{k-1} \| g_1^\xi \|^2 \leqslant c\tau^2 \, \|U\|^2_{H^2(0, T; L^2(\vartheta))} .$$

It is easy to prove that

$$\tau \sum_{\xi=0}^{k-1} \| g_2^\xi \|^2 \leqslant cN^{2-2\beta} \, \|U\|^2_{c(0, T; H^\beta(\vartheta))} .$$

By an argument similar to that of [14],

$$\tau \sum_{\xi=0}^{k-1} \| g_3^\xi \|^2 \leqslant cN^{2-2\beta} \, \|U\|^4_{c(0, T; H^\beta(\vartheta))} .$$

We have

$$\left|\tau \sum_{\xi=0}^{k-1} (g_4^\xi, e^\xi)\right| \leqslant \delta\tau \sum_{\xi=0}^{k-1} \left|\left(\frac{\partial W_t^\xi}{\partial x} - \frac{\partial^2 W(\xi\tau)}{\partial t\,\partial x}, \frac{\partial e^\xi}{\partial x}\right)\right|$$

$$\leqslant \delta\tau \sum_{\xi=0}^{k-1} \left(|e^\xi|_1^2 + \left\|\frac{\partial W_t^\xi}{\partial x} - \frac{\partial^2 W(\xi\tau)}{\partial t\,\partial x}\right\|\right)$$

$$\leqslant \delta\tau \sum_{\xi=0}^{k-1} |e^\xi|_1^2 + c\delta\tau^2 \,\|U\|^2_{H^2(0,T;H^1(\vartheta))},$$

$$\left|m\tau^2 \sum_{\xi=0}^{k-1} (g_4^\xi, e_t^\xi)\right| \leqslant \varepsilon\,\delta\tau^2 \sum_{\xi=0}^{k-1} |e_t^\xi|_1^2 + c\delta\tau^3 \,\|U\|^2_{H^2(0,T;H^1(\vartheta))}.$$

We have also

$$\tau \sum_{\xi=0}^{k-1} \|g_5^\xi\|^2 \leqslant cN^{2-2\beta} \,\|f\|^2_{c(0,T;H^{\beta-1}(\vartheta))},$$

$$\|e^0\|_1^2 \leqslant cN^{2-2\beta} \,\|U_0\|_\beta^2.$$

The previous estimations give

$$\|e^0\|_1^2 + \tau \sum_{\xi=0}^{k-1} \left(\sum_{l=1,2,3,5} \|g_l^\xi\|^2 + |(g_4^\xi, e^\xi + m\tau e_t^\xi)|\right)$$

$$\leqslant \delta\tau \sum_{\xi=0}^{k-1} (|e^\xi|_1^2 + \varepsilon\tau\,|e_t^\xi|_1^2) + c\tau^2 \,\|U\|^2_{H^2(0,T;L^2(\vartheta))} + c\tau^2 \,\|U\|^2_{H^2(0,T;H^1(\vartheta))}$$

$$+ cN^{2-2\beta} \,\|U\|^4_{c(0,T;H^\beta(\vartheta))} + cN^{2-2\beta} \,\|U_0\|_\beta^2 + cN^{2-2\beta} \,\|f\|^2_{c(0,T;H^{\beta-1}(\vartheta))}.$$

Finally we finish the proof with an argument similar to that in Section VI and with the use of the triangular inequality.

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